HL Paper 1

Part A. If z is a non-zero complex number, we define $L(z)$ by the equation [9]

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \leqslant \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \mathrm{i})$;

(iii) $L(-1 + \mathrm{i})$.

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2 .

Part B. Let f be a function with domain $\mathbb{R}$ that satisfies the conditions, [14]

$f(x + y) = f(x)f(y)$, for all x and y and $f(0) \neq 0$.

(a) Show that $f(0) = 1$.

(b) Prove that $f(x) \neq 0$, for all $x \in \mathbb{R}$.

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation

$f'(x) = k\,f(x)$, where $k = f'(0)$.

(d) Solve the differential equation to find an expression for $f(x)$.

Markscheme

Part A. (a) $|z| = z$, $\arg(z) = 0$ ***A1A1***

so $L(z) = \ln z$ ***AG N0***

[2 marks]

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$ ***A1A1 N2***

(ii) $L(1 - \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{7\pi}{4}$ ***A1A1 N2***

(iii) $L(-1 + \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{3\pi}{4}$ ***A1 N1***

[5 marks]

(c) for comparing the product of two of the above results with the third ***M1***

for stating the result $-1 + \mathrm{i} = -1 \times (1 - \mathrm{i})$ and $L(-1 + \mathrm{i}) \neq L(-1) + L(1 - \mathrm{i})$ ***R1***

hence, the property $L(z_1 z_2) = L(z_1) + L(z_2)$

does not hold for all values of z_1 and z_2 ***AG N0***

[2 marks]

Total [9 marks]

Part B. (a) from $f(x+y)=f(x)f(y)$

for $x=y=0$ ***M1***

we have $f(0+0)=f(0)f(0) \Leftrightarrow f(0)=(f(0))^2$ ***A1***

as $f(0)\neq 0$, this implies that $f(0)=1$ ***R1AG*** ***N0***

[3 marks]

(b) **METHOD 1**

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x) \Leftrightarrow f(0)=f(x)f(-x)$ ***M1A1***

as $f(0)\neq 0$ this implies that $f(x)\neq 0$ ***R1AG*** ***N0***

METHOD 2

suppose that, for a value of x, $f(x)=0$ ***M1***

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x) \Leftrightarrow f(0)=f(x)f(-x)$ ***A1***

substituting $f(x)$ by 0 gives $f(0)=0$ which contradicts part (a) ***R1***

therefore $f(x)\neq 0$ for all x. ***AG*** ***N0***

[3 marks]

(c) by the definition of derivative

$f'(x)=\lim\limits_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ ***(M1)***

$=\lim\limits_{h\to 0}\left(\frac{f(x)f(h)-f(x)f(0)}{h}\right)$ ***A1(A1)***

$=\lim\limits_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right)f(x)$ ***A1***

$=f'(0)f(x) \quad (=k\ f(x))$ ***AG*** ***N0***

[4 marks]

(d) $\int \frac{f'(x)}{f(x)}\mathrm{d}x=\int k\mathrm{d}x \Rightarrow \ln f(x)=kx+C$ ***M1A1***

$\ln f(0)=C \Rightarrow C=0$ ***A1***

$f(x)=\mathrm{e}^{kx}$ ***A1*** ***N1***

Note: Award ***M1A0A0A0*** if no arbitrary constant C .

[4 marks]

Total [14 marks]

Examiners report

Part A: Part A was answered well by a fair amount of candidates, with some making mistakes in calculating the arguments of complex numbers, as well as careless mistakes in finding the products of complex numbers.

Part B proved demanding for most candidates, particularly parts (c) and (d). A surprising number of candidates did not seem to know what was meant by the ‘definition of derivative’ in part (c) as they attempted to use quotient rule rather than first principles.

A gourmet chef is renowned for her spherical shaped soufflé. Once it is put in the oven, its volume increases at a rate proportional to its radius.

(a) Show that the radius r cm of the soufflé, at time t minutes after it has been put in the oven, satisfies the differential equation $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$, where k is a constant.

(b) Given that the radius of the soufflé is 8 cm when it goes in the oven, and 12 cm when it’s cooked 30 minutes later, find, to the nearest cm, its radius after 15 minutes in the oven.

Markscheme

(a) $\frac{dV}{\mathrm{d}t} = cr$ ***A1***

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1A1***

$\Rightarrow 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = cr$ ***M1***

$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{4\pi r}$ ***A1***

$= \frac{k}{r}$ ***AG***

[5 marks]

(b) $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$

$\Rightarrow \int r\mathrm{d}r = \int k\mathrm{d}t$ ***M1***

$\frac{r^2}{2} = kt + d$ ***A1***

An attempt to substitute either $t = 0, r = 8$ or $t = 30, r = 12$ ***M1***

When $t = 0, r = 8$

$\Rightarrow d = 32$ ***A1***

$\Rightarrow \frac{r^2}{2} = kt + 32$

When $t = 30, r = 12$

$\Rightarrow \frac{12^2}{2} = 30k + 32$

$\Rightarrow k = \frac{4}{3}$ ***A1***

$\therefore \frac{r^2}{2} = \frac{4}{3}t + 32$

When $t = 15$, $\frac{r^2}{2} = \frac{4}{3}15 + 32$ ***M1***

$\Rightarrow r^2 = 104$ ***A1***

$r \approx 10$ cm ***A1***

Note: Award ***M0*** to incorrect methods using proportionality which give solution $r = 10$ cm .

[8 marks]

Total [13 marks]

Examiners report

Candidates found this question quite difficult, with only the better students making appreciable progress on part (a). Relatively few candidates recognised that part (b) was asking them to solve a differential equation. Many students tried methods involving direct proportion, which did not lead anywhere.

Find y in terms of x, given that $(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2\tan y$ and $y = \frac{\pi}{2}$ when $x = 0$.

Markscheme

$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2\tan y \Rightarrow \int \frac{\mathrm{d}y}{\tan y} = \int \frac{2x^2}{1+x^3}\mathrm{d}x$ ***M1***

$\int \frac{\cos y}{\sin y}\mathrm{d}y = \frac{2}{3}\int \frac{3x^2}{1+x^3}\mathrm{d}x$ ***(A1)(A1)***

$\ln|\sin y| = \frac{2}{3}\ln\left|1+x^3\right| + C$ ***A1A1***

Notes: Do not penalize omission of modulus signs.

Do not penalize omission of constant at this stage.

EITHER

$\ln\left|\sin\frac{\pi}{2}\right| = \frac{2}{3}\ln|1| + C \Rightarrow C = 0$ ***M1***

OR

$|\sin y| = A\left|1+x^3\right|^{\frac{2}{3}},\ A = \mathrm{e}^C$

$\left|\sin\frac{\pi}{2}\right| = A\left|1+0^3\right|^{\frac{2}{3}} \Rightarrow A = 1$ ***M1***

THEN

$y = \arcsin\left((1+x^3)^{\frac{2}{3}}\right)$ ***A1***

Note: Award ***M0A0*** if constant omitted earlier.

[7 marks]

Examiners report

Many candidates separated the variables correctly but were then unable to perform the integrations.

A certain population can be modelled by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = k\,y\cos kt$, where y is the population at time t hours and k is a positive constant.

(a) Given that $y = y_0$ when $t = 0$, express y in terms of k , t and y_0 .

(b) Find the ratio of the minimum size of the population to the maximum size of the population.

Markscheme

(a) $\frac{\mathrm{d}y}{\mathrm{d}t} = ky\cos(kt)$

$\frac{\mathrm{d}y}{y} = k\cos(kt)\mathrm{d}t$ ***(M1)***

$\int \frac{\mathrm{d}y}{y} = \int k\cos(kt)\mathrm{d}t$ ***M1***

$\ln y = \sin(kt) + c$ ***A1***

$y = A\mathrm{e}^{\sin(kt)}$

$t = 0 \Rightarrow y_0 = A$ ***(M1)***

$\Rightarrow y = y_0\mathrm{e}^{\sin kt}$ ***A1***

(b) $-1 \leqslant \sin kt \leqslant 1$ ***(M1)***

$y_0\mathrm{e}^{-1} \leqslant y \leqslant y_0\mathrm{e}^{1}$

so the ratio is $\frac{1}{\mathrm{e}} : \mathrm{e}$ or $1 : \mathrm{e}^2$ ***A1***

[7 marks]

Examiners report

Part (a) was done successfully by many candidates. However, very few attempted part (b).

The curve C with equation $y = f(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2),\ y > 1,$$

and $y = \mathrm{e}$ when $x = 2$.

a. Find the equation of the tangent to C at the point (2, e). [3]

b. Find $f(x)$. [11]

c. Determine the largest possible domain of f. [6]

d. Show that the equation $f(x) = f'(x)$ has no solution. [4]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}}{\ln \mathrm{e}}(2+2) = 4\mathrm{e}$ ***A1***

at (2, e) the tangent line is $y - \mathrm{e} = 4\mathrm{e}(x-2)$ ***M1***

hence $y = 4\mathrm{e}x - 7\mathrm{e}$ ***A1***

[3 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}\mathrm{d}y = (x+2)\mathrm{d}x$ ***M1***

$\int \frac{\ln y}{y}\mathrm{d}y = \int (x+2)\mathrm{d}x$

using substitution $u = \ln y;\ \mathrm{d}u = \frac{1}{y}\mathrm{d}y$ ***(M1)(A1)***

$\Rightarrow \int \frac{\ln y}{y}\mathrm{d}y = \int u\mathrm{d}u = \frac{1}{2}u^2$ ***(A1)***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ ***A1A1***

at (2, e), $\frac{(\ln \mathrm{e})^2}{2} = 6 + c$ ***M1***

$\Rightarrow c = -\frac{11}{2}$ ***A1***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2+4x-11} \Rightarrow y = \mathrm{e}^{\pm\sqrt{x^2+4x-11}}$ ***M1A1***

since y > 1, $f(x) = \mathrm{e}^{\sqrt{x^2+4x-11}}$ ***R1***

Note:*M1* for attempt to make *y* the subject.

[11 marks]

c. **EITHER**

$x^2 + 4x - 11 > 0$ ***A1***

using the quadratic formula ***M1***

critical values are $\frac{-4\pm\sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$ ***A1***

using a sign diagram or algebraic solution ***M1***

$x < -2 - \sqrt{15};\ x > -2 + \sqrt{15}$ ***A1A1***

OR

$x^2 + 4x - 11 > 0$ ***A1***

by methods of completing the square ***M1***

$(x+2)^2 > 15$ ***A1***

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$ ***(M1)***

$x < -2 - \sqrt{15};\ x > -2 + \sqrt{15}$ ***A1A1***

[6 marks]

d. $f(x) = f'(x) \Rightarrow f(x) = \frac{f(x)}{\ln f(x)}(x+2)$ ***M1***

$\Rightarrow \ln(f(x)) = x + 2$ $\left(\Rightarrow x + 2 = \sqrt{x^2+4x-11}\right)$ ***A1***

$\Rightarrow (x+2)^2 = x^2 + 4x - 11 \Rightarrow x^2 + 4x + 4 = x^2 + 4x - 11$ ***A1***

$\Rightarrow 4 = -11,$ hence $f(x) \neq f'(x)$ ***R1AG***

[4 marks]

Examiners report

a. Nearly always correctly answered.

b. Most candidates separated the variables and attempted the integrals. Very few candidates made use of the condition $y > 1$, so losing 2 marks.

c. Part (c) was often well answered, sometimes with follow through.

d. Only the best candidates were successful on part (d).